Probability density of the determinant of a random Hermitian matrix

1998 J. Phys. A: Math. Gen. 31 5377

(http://iopscience.iop.org/0305-4470/31/23/018)

J. Phys. A: Math. Gen. **31** (1998) 5377–5391. Printed in the UK PII: S0305-4470(98)90281-5

Probability density of the determinant of a random Hermitian matrix

Madan Lal Mehta† and Jean-Marie Normand‡
CEA/Saclay, Service de Physique Théorique, F-91191 Gif-sur-Yvette Cedex, France

Received 5 January 1998, in final form 11 March 1998

Abstract. The probability density function for the determinant of a $n \times n$ random Hermitian matrix taken from the Gaussian unitary ensemble is calculated. It is found to be a Meijer G-function or a linear combination of two Meijer G-functions, depending on the parity of n. The integer moments of this probability density are also given.

1. Introduction and results

Three ensembles of random matrices have been studied extensively [1]. These are the ensembles of matrices A defined by the probability density $\propto \exp(-\operatorname{tr} A^2)$ of the matrix elements. The matrix A is real symmetric for the Gaussian orthogonal ensemble, it is complex Hermitian for the Gaussian unitary ensemble and it is quaternion self-dual for the Gaussian symplectic ensemble. The question concerning the distribution of the determinant was asked only once by Wigner [2], who gave the mean value of the logarithm of the $n \times n$ determinant $[\delta_{jk} + \varepsilon_{jk}]$ in the limit $n \to \infty$ with ε_{jk} ($\varepsilon_{jk} = \varepsilon_{kj}$, if real and $\varepsilon_{jk} = \varepsilon^*_{kj}$, if complex) of order $1/n$. For a random matrix taken from the Gaussian unitary ensemble we shall calculate the probability density of its determinant. The same question concerning the other two ensembles (orthogonal and symplectic) remains open.

Our method consists of calculating explicitly, in section 2 the Mellin transforms of the even and odd parts of the probability density $g_n(y)$, equation (3.1), of the determinant $y = x_1 x_2 \ldots x_n$ where $x_1, \ldots, x_n$ are the eigenvalues of the random matrix. For this calculation we apply a method used recently [3] to compute the expectation value of any function of eigenvalues of the form $\prod_{j=1}^n \phi(x_j)$. We then use the inverse Mellin transform in section 3. Our main results for positive integers n are as follows. They have a different form accordingly as n is odd or even, and for clarity we present them separately.

$$g_{2m+1}(y) = N_{2m+1} G^{2m+1,0}_{0,2m+1}(y^2|0, 1, 1, 2, 2, \ldots, m, m) \tag{1.1}$$

$$g_{2m}(y) = N_{2m}[G^{2m,0}_{0,2m}(y^2|0, 1, 1, 2, 2, \ldots, m-1, m-1, m) \\ +(-1)^m \operatorname{sign}(y) G^{2m,0}_{0,2m}(y^2|\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{3}{2}, \ldots, m-\tfrac{1}{2}, m-\tfrac{1}{2})] \tag{1.2}$$

$$N_{2m+1} := 2^{m(2m+1)} \pi^{-(m+1/2)} \prod_{j=0}^{m} j! \Bigg/ \prod_{j=m}^{2m} j! \tag{1.3}$$

† Member of CNRS, France. E-mail address: mehta@spht.saclay.cea.fr
‡ E-mail address: norjm@spht.saclay.cea.fr

$$N_{2m} := 2^{m(2m-1)}\pi^{-m}\prod_{j=0}^{m-1} j! \Big/ \prod_{j=m}^{2m-1} j!. \tag{1.4}$$

Here $G_{0,n}^{n,0}$ is a Meijer G-function [4].

For $y = 0$, one obtains in section 4,

$$g_{2m+1}(0) = 2^{m(2m+1)}\pi^{-(m+1/2)}\prod_{j=0}^{m} j!^3 \Big/ \left(m!^2 \prod_{j=m}^{2m} j!\right) \tag{1.5}$$

$$g_{2m}(0) = 2^{m(2m-1)}\pi^{-m}\prod_{j=0}^{m-1} j!^3 \Big/ \prod_{j=m-1}^{2m-1} j! \tag{1.6}$$

$$g'_{2m+1}(0) = 0 \tag{1.7}$$

$$g'_{2m}(0) = (-1)^m \times \infty. \tag{1.8}$$

In the vicinity of $y = 0$, one has the expansion of $g_n(y)$ as

$$g_1(y) = \frac{1}{\sqrt{\pi}}[1 - y^2 + \mathrm{O}(y^4)] \tag{1.9}$$

$$g_2(y) = \frac{2}{\pi}[1 + 2y\ln|y| + 2\gamma y + 2y^2\ln|y| + (2\gamma - 1)y^2 + \mathrm{O}(y^3\ln|y|)] \tag{1.10}$$

$$g_{2m+1}(y) = g_{2m+1}(0)\left[1 - \frac{2y^2}{(m-1)!^2}(\ln^2|y| + \alpha_m\ln|y| + \beta_m) + \mathrm{O}(y^4\ln^4|y|)\right] \tag{1.11}$$

$$\begin{aligned} g_{2m}(y) = g_{2m}(0)\Bigg\{&1 - (-1)^m\frac{2y}{(m-1)!}\left[\ln|y| + \gamma - \sum_{j=1}^{m-1}\psi(j)\right] \\ &- \frac{2y^2}{(m-1)!(m-2)!}(\ln^2|y| + \alpha'_m\ln|y| + \beta'_m) + \mathrm{O}(y^3\ln^3|y|)\Bigg\} \end{aligned} \tag{1.12}$$

where

$$\alpha_m = 3\gamma - 1 - 2\sum_{j=1}^{m-1}\psi(j) \tag{1.13}$$

$$\alpha'_m = 3\gamma - 1 - \psi(m-1) - 2\sum_{j=1}^{m-2}\psi(j) \tag{1.14}$$

$$\beta_m = \frac{1}{4}\left[\alpha_m^2 + 1 + \frac{\pi^2}{2} + 2\sum_{j=1}^{m-1}\psi'(j)\right] \tag{1.15}$$

$$\beta'_m = \frac{1}{4}\left[\alpha'^2_m + 1 + \frac{\pi^2}{2} + \psi'(m-1) + 2\sum_{j=1}^{m-2}\psi'(j)\right]. \tag{1.16}$$

In these equations, $\psi(z) := \Gamma'(z)/\Gamma(z)$ is the psi function [5] and ψ' is its derivative, $\gamma = -\psi(1) \approx 0.5772$ is the Euler constant and $\psi(j)$ and $\psi'(j)$ for $j \leqslant 0$ are interpreted to be zero.

For large $|y|$, the asymptotic expressions for $g_n(y)$ are derived in section 5.

$$\begin{aligned} g_{2m+1}(y) = {}& 2^{2m(m+1)}[\pi(2m+1)]^{-1/2}\left(\prod_{j=0}^{m} j! \Big/ \prod_{j=m}^{2m} j!\right)\mathrm{e}^{-(2m+1)|y|^{2/(2m+1)}}|y|^{2m^2/(2m+1)} \\ &\times\left[1 + \frac{m^2(m+1)^2}{6(2m+1)}|y|^{-2/(2m+1)} + \mathrm{O}(|y|^{-4/(2m+1)})\right] \qquad |y| \gg 1 \end{aligned} \tag{1.17}$$

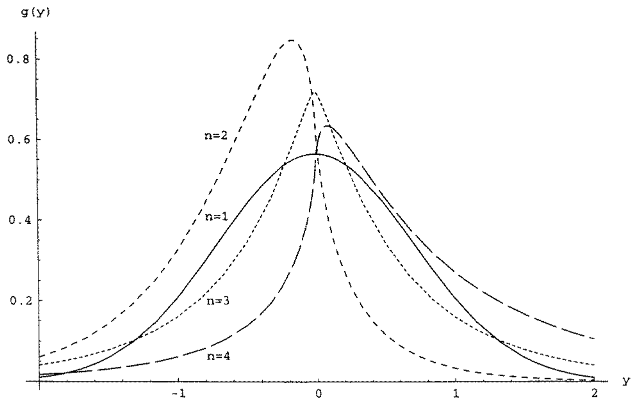

Figure 1. Graphs of the functions $g_n(y)$ for $n = 1, 2, 3$ and 4.

$$g_{2m}(y) = 2^{2m^2-3}\left(\frac{m}{\pi}\right)^{1/2}\left(\prod_{j=0}^{m-1} j! \Big/ \prod_{j=m}^{2m-1} j!\right) \mathrm{e}^{-2m|y|^{1/m}} |y|^{m-1-1/(2m)} [1 + \mathrm{O}(|y|^{-1/m})] \tag{1.18}$$

if either m is odd and $y \gg 1$, or m is even and $y \ll -1$

$$g_{2m}(y) = 2^{2m^2}(\pi m)^{-1/2}\left(\prod_{j=0}^{m-1} j! \Big/ \prod_{j=m}^{2m-1} j!\right) \mathrm{e}^{-2m|y|^{1/m}} |y|^{m-1+1/(2m)} \times \left[1 + \frac{4m^4 - 2m^2 + 1}{48m}|y|^{-1/m} + \mathrm{O}(|y|^{-2/m})\right] \tag{1.19}$$

if either m is even and $y \gg 1$, or m is odd and $y \ll -1$.

The particular cases $n = 1, 2, 3$ and 4 are studied in appendix A.1 and the corresponding functions $g_n(y)$ are plotted in figure 1.

The moments of the probability density $g_n(y)$ for $q = 1, 2, \ldots$

$$M(n, q) := \int_{-\infty}^{\infty} g_n(y) y^q \, \mathrm{d}y \tag{1.20}$$

are deduced in section 6

$$M(2m + 1, 2p + 1) = 0 \tag{1.21}$$

$$M(2m + 1, 2p) = \frac{\Gamma(p + \frac{1}{2})}{\Gamma(\frac{1}{2})} \prod_{j=1}^{m} \left[\frac{\Gamma(p + j + \frac{1}{2})}{\Gamma(j + \frac{1}{2})}\right]^2 \tag{1.22}$$

$$M(2m, 2p + 1) = (-1)^m \frac{\Gamma(\frac{1}{2})}{\Gamma(m + \frac{1}{2})} \prod_{j=0}^{m-1} \left[\frac{\Gamma(p + j + \frac{3}{2})}{\Gamma(j + \frac{1}{2})}\right]^2 \tag{1.23}$$

$$M(2m, 2p) = \frac{\Gamma(p + m + \frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(p + \frac{1}{2})\Gamma(m + \frac{1}{2})} \prod_{j=0}^{m-1} \left[\frac{\Gamma(p + j + \frac{1}{2})}{\Gamma(j + \frac{1}{2})}\right]^2. \tag{1.24}$$

In appendix A.3 we make some remarks concerning the probability density of the determinant of a matrix taken from the Gaussian symplectic and Gaussian orthogonal ensembles, in appendix A.4 concerning complex matrices and in appendix A.5 concerning quaternion real matrices.

2. Calculation of Mellin transforms

For p Hermitian $n \times n$ matrices coupled in a chain we recently used [3] a method of computing the average of a function of the form $\prod_{\mu=1}^{p}\prod_{k=1}^{n}\phi_\mu(x_{\mu k})$, where $x_{\mu k}$, $k = 1, \ldots, n$, are the eigenvalues of the μth matrix in the chain. We took $\phi_\mu(x)$ to be $1 - \chi_\mu(x)$, where χ_μ is the characteristic function of the interval I_μ and obtained the probability that the interval I_μ does not contain any eigenvalue of the μth matrix in the chain. We now apply the same method to compute the Mellin transforms of the even and odd parts

$$g_n^{\pm}(y) := \tfrac{1}{2}[g_n(y) \pm g_n(-y)] \tag{2.1}$$

of the probability density $g_n(y)$ of the determinant of one matrix. The probability density of the eigenvalues $\boldsymbol{x} := \{x_1, x_2, \ldots, x_n\}$ of a random matrix taken from the Gaussian unitary ensemble for $n = 1, 2, \ldots$ is [1]

$$F(\boldsymbol{x}) := C_n \exp\left(-\sum_{j=1}^{n} x_j^2\right)\Delta^2(\boldsymbol{x}) \tag{2.2}$$

$$\Delta(\boldsymbol{x}) := \begin{cases} 1 & n = 1 \\ \prod_{1\leqslant j<k\leqslant n}(x_k - x_j) & n = 2, 3, \ldots \end{cases} \tag{2.3}$$

$$C_n := 2^{n(n-1)/2}\pi^{-n/2} \Big/ \prod_{j=1}^{n} j!. \tag{2.4}$$

Recall that a polynomial is called monic if the coefficient of its highest power is one. We write $\Delta(\boldsymbol{x})$ as a polynomial alternant,

$$\Delta^2(\boldsymbol{x}) = (\det[x_j^{i-1}])^2 = \det[P_{i-1}(x_j)]\det[Q_{i-1}(x_j)]_{i,j=1,\ldots,n} \tag{2.5}$$

where $P_i(x)$ and $Q_i(x)$ are any monic polynomials of degree i. Expanding the determinants one has

$$\Delta^2(\boldsymbol{x}) = \sum_{(i)}\sum_{(j)} \sigma(i)\sigma(j)P_{i_1}(x_1)\ldots P_{i_n}(x_n)Q_{j_1}(x_1)\ldots Q_{j_n}(x_n) \tag{2.6}$$

where $\sigma(i)$ is the sign of the permutation $(i) := \binom{0,\ldots,n-1}{i_1,\ldots,i_n}$, the sum (i) is over all the $n!$ permutations (i), and similarly for the permutations (j).

If $\Phi(\boldsymbol{x}) = \prod_{j=1}^{n}\phi(x_j)$, then the average value of $\Phi(\boldsymbol{x})$ is

$$\begin{aligned}\langle\Phi(\boldsymbol{x})\rangle &:= \int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} F(\boldsymbol{x})\Phi(\boldsymbol{x})\,\mathrm{d}x_1\ldots\mathrm{d}x_n \\ &= C_n\sum_{(i)}\sum_{(j)}\sigma(i)\sigma(j)\prod_{k=1}^{n}\int_{-\infty}^{\infty} P_{i_k}(x_k)Q_{j_k}(x_k)\mathrm{e}^{-x_k^2}\phi(x_k)\,\mathrm{d}x_k \\ &= C_n\sum_{(i)}\sum_{(j)}\sigma(i)\sigma(j)\Phi_{i_1,j_1}\ldots\Phi_{i_n,j_n} \\ &= C_n n!\det[\Phi_{i,j}]_{i,j=0,\ldots,n-1}\end{aligned} \tag{2.7}$$

where

$$\Phi_{i,j} := \int_{-\infty}^{\infty} P_i(x)Q_j(x)\mathrm{e}^{-x^2}\phi(x)\,\mathrm{d}x. \tag{2.8}$$

The determinant of the matrix is $y = x_1 \dots x_n$ with a probability density $g_n(y)$. The Mellin transform of the even part of $g_n(y)$ is

$$\mathcal{M}_n^+(s) := \int_0^\infty y^{s-1} g_n^+(y)\,\mathrm{d}y \tag{2.9}$$

$$= \tfrac{1}{2}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} F(\boldsymbol{x})|x_1\dots x_n|^{s-1}\,\mathrm{d}x_1\dots\mathrm{d}x_n \tag{2.10}$$

$$= \tfrac{1}{2}\langle |x_1\dots x_n|^{s-1}\rangle \tag{2.11}$$

$$= \tfrac{1}{2}C_n n!\det[\Phi_{i,j}^+]_{i,j=0,\dots,n-1} \tag{2.12}$$

where with $\phi(x) = |x|^{s-1}$ in equation (2.8), one has

$$\Phi_{i,j}^+ := \int_{-\infty}^{\infty} P_i(x)Q_j(x)\mathrm{e}^{-x^2}|x|^{s-1}\,\mathrm{d}x. \tag{2.13}$$

We can choose $P_i(x)$ and $Q_i(x)$ as any monic polynomials of degree i. They can be chosen to make the matrix $[\Phi_{i,j}^+]$ diagonal. However, let us take $P_i(x) = Q_i(x) = x^i$. Then

$$\Phi_{i,j}^+ := \int_{-\infty}^{\infty} x^{i+j}|x|^{s-1}\mathrm{e}^{-x^2}\,\mathrm{d}x \qquad \mathrm{Re}\, s > 0$$

$$= \begin{cases} \Gamma[(s+i+j)/2] & i+j \text{ even} \\ 0 & i+j \text{ odd.} \end{cases} \tag{2.14}$$

The alternate elements of the $n \times n$ determinant $[\Phi_{i,j}^+]$ being zero, we can rearrange its rows and columns so as to collect the zero elements separate from the non-zero elements. Thus

$$\det[\Phi_{i,j}^+]_{i,j=0,\dots,n-1} = \det[\Phi_{2i,2j}^+]_{i,j=0,\dots,[(n-1)/2]}\det[\Phi_{2i+1,2j+1}^+]_{i,j=0,\dots,[(n-2)/2]} \tag{2.15}$$

where $[x]$ denotes the largest integer less than or equal to x. It is straightforward to evaluate the determinants $[\Phi_{2i,2j}^{(+)}]$ and $[\Phi_{2i+1,2j+1}^+]$ (cf appendix A.2)

$$\det[\Phi_{2i,2j}^+]_{i,j=0,\dots,[(n-1)/2]} = \prod_{j=0}^{[(n+1)/2]}\left[j!\,\Gamma\left(\frac{s}{2}+j\right)\right] \tag{2.16}$$

$$\det[\Phi_{2i+1,2j+1}^+]_{i,j=0,\dots,[n/2]} = \prod_{j=0}^{[n/2]}\left[j!\,\Gamma\left(\frac{s}{2}+j+1\right)\right]. \tag{2.17}$$

Putting in the constants, we find

$$\mathcal{M}_n^+(s) = \frac{1}{2}N_n\prod_{j=1}^{n}\Gamma\left(\frac{s}{2}+b_j^+\right) \qquad \mathrm{Re}\, s > 0 \tag{2.18}$$

where

$$b_j^+ := [j/2] \qquad j = 1, 2, \dots. \tag{2.19}$$

Note that, from the normalization of $g_n(y)$, one has

$$1 = \int_{-\infty}^{\infty} g_n(y)\,\mathrm{d}y = 2\int_0^\infty g_n^+(y)\,\mathrm{d}y \tag{2.20}$$

therefore

$$N_n^{-1} = \prod_{j=1}^{n}\Gamma\left(\tfrac{1}{2}+b_j^+\right). \tag{2.21}$$

Similarly the Mellin transform of the odd part of $g_n(y)$ is

$$\mathcal{M}_n^-(s) := \int_0^\infty y^{s-1} g_n^-(y)\,\mathrm{d}y \tag{2.22}$$

$$= \tfrac{1}{2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} F(\boldsymbol{x})\,\mathrm{sign}(x_1\ldots x_n)|x_1\ldots x_n|^{s-1}\,\mathrm{d}x_1\ldots\mathrm{d}x_n \tag{2.23}$$

$$= \tfrac{1}{2}\langle \mathrm{sign}(x_1\ldots x_n)|x_1\ldots x_n|^{s-1}\rangle \tag{2.24}$$

$$= \tfrac{1}{2} C_n n!\,\det[\Phi^-_{i,j}]_{i,j=0,\ldots,n-1} \tag{2.25}$$

where with $\phi(x) = \mathrm{sign}(x)|x|^{s-1}$ in equation (2.8), one has

$$\Phi^-_{i,j} := \int_{-\infty}^{\infty} x^{i+j}\,\mathrm{sign}(x)|x|^{s-1}\mathrm{e}^{-x^2}\,\mathrm{d}x \qquad \mathrm{Re}\, s > 0$$
$$= \begin{cases} \Gamma[(s+i+j)/2] & i+j \text{ odd} \\ 0 & i+j \text{ even.} \end{cases} \tag{2.26}$$

In the $n\times n$ determinant $[\Phi^-_{i,j}]$, the zero and non-zero elements can again be separated and the two resulting determinants computed. One has

$$\det[\Phi^-_{i,j}] = (-1)^{n/2}\{\det[\Phi^-_{2i,2j+1}]_{i,j=0,\ldots,[n/2]}\}^2$$
$$= \begin{cases} (-1)^{n/2}\displaystyle\prod_{j=0}^{n/2-1}\left[j!\,\Gamma\left(\frac{s+1}{2}+j\right)\right]^2 & n \text{ even} \\ 0 & n \text{ odd.} \end{cases} \tag{2.27}$$

So that the Mellin transform of the odd part of $g_n(y)$ is

$$\mathcal{M}_n^-(s) = \begin{cases} (-1)^{n/2}\dfrac{1}{2}N_n\displaystyle\prod_{j=1}^{n}\Gamma\left(\frac{s}{2}+b_j^-\right) & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \qquad \mathrm{Re}\, s > 0 \tag{2.28}$$

where

$$b_j^- := [(j-1)/2] + \tfrac{1}{2} \qquad j = 1, 2, \ldots. \tag{2.29}$$

3. Inverse Mellin transforms

The probability density $g_n(y)$ is defined using the Dirac delta function by

$$g_n(y) := \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} F(\boldsymbol{x})\delta(y - x_1\ldots x_n)\,\mathrm{d}x_1\ldots\mathrm{d}x_n. \tag{3.1}$$

From equation (2.2) an integration over x_n gives

$$g_n(y) = C_n\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left[-\sum_{j=1}^{n-1}x_j^2 - \frac{y^2}{(x_1\ldots x_{n-1})^2}\right]\frac{1}{|x_1\ldots x_{n-1}|}$$
$$\times\prod_{j=1}^{n-1}\left(x_j - \frac{y}{x_1\ldots x_{n-1}}\right)^2\prod_{1\leqslant i<j\leqslant n-1}(x_i - x_j)^2. \tag{3.2}$$

The integrand in the above equation is the product of an exponential term by a polynomial in $y, x_1, \ldots, x_{n-1}$ and divided by $|x_1\ldots x_{n-1}|^{2n-1}$. When any x_j goes either to infinity or to zero the integrand has a decreasing Gaussian factor. It follows that the integral (3.2) is convergent. Furthermore, the integrand is clearly a continuous function of y. Therefore, from equation (3.1) the probability density $g_n(y)$ is a continuous bounded function for any real y. From equation (2.1) the even and odd parts $g_n^{\pm}(y)$ of $g_n(y)$ are also bounded and continuous. Their Mellin transforms $\mathcal{M}_n^{\pm}(s)$ are analytic in the right half complex s-plane $\mathrm{Re}\, s > 0$. Thus they are uniquely determined by the inverse Mellin transform of $\mathcal{M}_n^{\pm}(s)$ [6].

Looking at the tables of integral transforms [7] one finds that the Mellin transform of a Meijer G-function is the ratio of products of gamma functions. In particular [7]

$$\int_0^{\infty} x^{s-1} G_{0,n}^{n,0}(x|b_1, \ldots, b_n)\, \mathrm{d}x = \prod_{j=1}^{n} \Gamma(s + b_j). \tag{3.3}$$

By a change of x into y^2 and s into $s/2$, this can be written as

$$2\int_0^{\infty} y^{s-1} G_{0,n}^{n,0}(y^2|b_1, \ldots, b_n)\, \mathrm{d}y = \prod_{j=1}^{n} \Gamma\left(\frac{s}{2} + b_j\right). \tag{3.4}$$

Comparing this last equation with equations (2.18) and (2.28), one obtains for $y \geqslant 0$

$$g_n^{+}(y) = N_n G_{0,n}^{n,0}(y^2|b_1^{+}, \ldots, b_n^{+}) \tag{3.5}$$

$$g_n^{-}(y) = \begin{cases} (-1)^{n/2} N_n G_{0,n}^{n,0}(y^2|b_1^{-}, \ldots, b_n^{-}) & n \text{ even} \\ 0 & n \text{ odd.} \end{cases} \tag{3.6}$$

Taking into account the symmetry properties of $g_n^{\pm}(y)$, one obtains for all real y the results announced in equations (1.1) and (1.2).

4. Computation of the G-functions and behaviour near the origin

The G-functions have convergent series expansions which are convenient for their numerical evaluation. By definition [4]

$$G_{0,n}^{n,0}(y^2|b_1^{\pm}, \ldots, b_n^{\pm}) = \frac{1}{2\mathrm{i}\pi} \int_{\mathcal{L}} y^{2s} \prod_{j=1}^{n} \Gamma(b_j^{\pm} - s)\, \mathrm{d}s. \tag{4.1}$$

The contour $\mathcal{L}$ goes from $-\mathrm{i}\infty$ to $+\mathrm{i}\infty$ so that all poles of $\Gamma(b_j^{\pm} - s)$ lie to the right of the path. It can be closed in the right half complex s-plane, so that the G-function is the negative of the sum of residues at its poles which from equations (2.19) and (2.29) all lie on the non-negative real axis.

When $n = 1$, $G_{0,1}^{1,0}(y^2|0)$ has simple poles at $j = 0, 1, \ldots$ with the residue $(-1)^{j+1} y^{2j}/j!$ and

$$G_{0,1}^{1,0}(y^2|0) = \mathrm{e}^{-y^2}. \tag{4.2}$$

For $n = 2$, one can either calculate the residues at the poles or consult the literature [8] to find that

$$G_{0,2}^{2,0}(y^2|b_1, b_2) = 2|y|^{b_1+b_2} K_{|b_2-b_1|}(2|y|) \tag{4.3}$$

with K_ν the modified Bessel function [9]. Thus

$$G_{0,2}^{2,0}(y^2|0, 1) = 2|y| K_1(2|y|) \qquad G_{0,2}^{2,0}(y^2|\tfrac{1}{2}, \tfrac{1}{2}) = 2|y| K_0(2|y|). \tag{4.4}$$

For $n \geqslant 2$, in the case of the $\{b_j^{+}\}$, $s = 0$ is a simple pole with residue $-\prod_{j=1}^{n} (b_j^{+})!$; $s = 1$ is a pole of order 2 or 3 according as $n = 2$ or $n \geqslant 3$; $s = 2$ is a pole of order

$\min(n, 5)$; $s = 3$ is a pole of order $\min(n, 7)$; and so on. It is straightforward to calculate the residue at $s = j$, by writing

$$\Gamma(k-s) = [(k-s)(k+1-s)\ldots(j-s)]^{-1}\Gamma(j+1-s) \qquad 0 \leqslant k \leqslant j. \tag{4.5}$$

The result is a series

$$G^{n,0}_{0,n}(y^2|b_1^+, \ldots, b_n^+) = 1 + \sum_{j=1}^{\infty} c_+(n, j, \ln|y|)y^{2j} \Bigg/ \prod_{k=1}^{\min(2j,n)} (j - b_k^+)! \tag{4.6}$$

where $c_+(n, j, \ln|y|)$ is a polynomial of order at most $n-1$ in $\ln|y|$. Similarly, in the case of $\{b_j^-\}$, $s = j + \frac{1}{2}$ is a pole of order $\min(n, 2j+2)$, calculation of the residue is again straightforward, and

$$G^{n,0}_{0,n}(y^2|b_1^-, \ldots, b_n^-) = \sum_{j=0}^{\infty} c_-(n, j, \ln|y|)|y|^{2j+1} \Bigg/ \prod_{k=1}^{\min(2j,n)} (j + \tfrac{1}{2} - b_k^-)! \tag{4.7}$$

where $c_-(n, j, \ln|y|)$ is again a polynomial of order at most $n-1$ in $\ln|y|$. For large j, owing to the presence of n factorials in the denominator, the convergence of the series (4.6) or (4.7) is better, the larger n is.

As an illustration we compute in appendix A.1 the functions $G^{n,0}_{0,n}(y^2|b_1^{\pm}, \ldots, b_n^{\pm},)$ for $n = 3$ and $n = 4$. We also give the expressions of $g_n(y)$ for $n = 1$, 2, 3 and 4, and plot their graphs in figure 1.

Near $s = 0$ the poles at $s = 0$ and $s = 1$ (resp. $s = \frac{1}{2}$) are dominant for the case $\{b_j^+\}$ (resp. $\{b_j^-\}$). Using equations (1.1) and (1.2) we find the expansions (1.10)–(1.12) for $g_n(y)$ near the origin with $n = 2, 3, \ldots$.

5. *G*-functions for large values of the variable

When $|x| \to \infty$, $|\arg x| \leqslant (n+1)\pi - \delta$, $\delta > 0$, we have the asymptotic expansion [10]

$$G^{n,0}_{0,n}(x|b_1^{\pm}, \ldots, b_n^{\pm}) = (2\pi)^{(n-1)/2} n^{-1/2} \exp(-nx^{1/n}) x^{\theta_n^{\pm}} [1 + c_n^{\pm} x^{-1/n} + \mathrm{O}(x^{-2/n})] \tag{5.1}$$

where

$$\theta_n^{\pm} = \frac{1}{n}\left(\sum_{j=1}^{n} b_j^{\pm} - \frac{n-1}{2}\right) \tag{5.2}$$

$$c_n^{\pm} = \frac{1}{2}\sum_{j=1}^{n}(b_j^{\pm})^2 - \frac{1}{2n}\left(\sum_{j=1}^{n} b_j^{\pm}\right)^2 - \frac{n^2-1}{24n}. \tag{5.3}$$

From the expressions of the b_j^+, equation (2.19), one has for n odd,

$$\theta_n^+ = \frac{(n-1)^2}{4n} \tag{5.4}$$

$$c_n^+ = \frac{(n^2-1)^2}{96n}. \tag{5.5}$$

Setting $n = 2m+1$, one obtains the asymptotic expression (1.17).

Similarly, for n even, equations (2.19) and (2.29) give

$$\theta_n^{\pm} = \frac{n^2 - 2n + 2}{4n} \tag{5.6}$$

$$c_n^+ + c_n^- = \frac{n^4 - 2n^2 + 4}{48n} \tag{5.7}$$

$$c_n^+ - c_n^- = \frac{n}{8}. \tag{5.8}$$

Setting $n = 2m$, one obtains the asymptotic expressions (1.18) and (1.19) according to the value of $(-1)^m \operatorname{sign}(y)$.

6. Integer moments of $g_n(y)$

The moments of the probability density $g_n(y)$ can be expressed in terms of the Mellin transforms $\mathcal{M}_n^{\pm}$ as follows. For $q = 0, 1, \ldots$

$$M(n,q) = [1 + (-1)^q] \int_0^\infty g_n^+(y) y^q \, \mathrm{d}y + [1 - (-1)^q] \int_0^\infty g_n^-(y) y^q \, \mathrm{d}y$$
$$= [1 + (-1)^q] \mathcal{M}_n^+(q+1) + [1 - (-1)^q] \mathcal{M}_n^-(q+1). \quad (6.1)$$

Replacing $\mathcal{M}_n^{\pm}(q+1)$ by their expressions, equations (2.18), (2.19), (2.28) and (2.29) yields on simplification values given in equations (1.21)–(1.24).

Acknowledgments

Among our colleagues H Navelet introduced us to the Meijer G-functions, R Balian suggested treating the particular case of $n = 2$ thus making us realize that the even and odd parts of $g_n(y)$ may behave differently, M Bergère suggested that the inverse Mellin transform was unique and the insistence of G Mahoux on mathematical rigour forced us to understand the complex analysis a little better. We are grateful to them all.

We thank the referee for bringing to our attention the book *Theory of Random Determinants* by V L Girko [13]. The questions raised or answered there do not include the ones considered here. However, this book provided us with further references on the subject. In particular, [14] gives the expressions for integer moments and the probability density of the determinant of a real square matrix without symmetry when the matrix elements are independent Gaussian random variables. In this case, since the probability density of the eigenvalues is not known, our method cannot be used.

Appendix

A.1. Particular cases n = 1, 2, 3 and 4

The relation between the G-functions and the $g_n(y)$ are given in equations (1.1)–(1.4). For $n = 1$ and 2, equations (4.2) and (4.4) yield

$$g_1(y) = \frac{1}{\sqrt{\pi}} \mathrm{e}^{-y^2} \quad (A.1)$$

$$g_2(y) = \frac{4}{\pi} [|y| K_1(2|y|) - y K_0(2|y|)]. \quad (A.2)$$

For $n = 3$ and $n = 4$ the expressions of the G-functions are not found in the literature and one has to calculate the residues at the poles as explained in section 4. As the order of any pole is not greater than 3 or 4, the calculation is not prohibitive. The relevant results are

$$G_{0,3}^{3,0}(y^2|0,1,1) = 1 + \frac{1}{2} \sum_{j=1}^{\infty} \frac{(-)^j j^2 y^{2j}}{j!^2}$$
$$\times \left\{ \left[2 \ln|y| + 3\gamma + \frac{2}{j} - 3S_1(j) \right]^2 + \frac{\pi^2}{2} - \frac{2}{j^2} + 3S_2(j) \right\} \quad (A.3)$$

$$G_{0,4}^{4,0}(y^2|0,1,1,2) = 1 - \frac{y^2}{2}\left[(2\ln|y| + 4\gamma - 1)^2 + \frac{2\pi^2}{3} + 1\right]$$
$$-\frac{1}{6}\sum_{j=2}^{\infty}\frac{j^3(j-1)y^{2j}}{j!^4}\left\{\left[2\ln|y| + 4\gamma + \frac{3}{j} + \frac{1}{j-1} - 4S_1(j)\right]^3\right.$$
$$+3\left[2\ln|y| + 4\gamma + \frac{3}{j} + \frac{1}{j-1} - 4S_1(j)\right]$$
$$\left.\times\left[\frac{2\pi^2}{3} - \frac{3}{j^2} - \frac{1}{(j-1)^2} + 4S_2(j)\right] + 8\zeta(3) + \frac{6}{j^3} + \frac{2}{(j-1)^3} - 8S_3(j)\right\} \tag{A.4}$$

$$G_{0,4}^{4,0}(y^2|\tfrac{1}{2},\tfrac{1}{2},\tfrac{3}{2},\tfrac{3}{2}) = -(2\ln|y| + 4\gamma)|y| - \frac{1}{6}\sum_{j=1}^{\infty}\frac{j^2|y|^{2j+1}}{j!^4}$$
$$\times\left\{\left[2\ln|y| + 4\gamma + \frac{2}{j} - 4S_1(j)\right]^3 + 3\left[2\ln|y| + 4\gamma + \frac{2}{j} - 4S_1(j)\right]\right.$$
$$\left.\times\left[\frac{2\pi^2}{3} - \frac{2}{j^2} + 4S_2(j)\right] + 8\zeta(3) + \frac{4}{j^3} - 8S_3(j)\right\} \tag{A.5}$$

where

$$S_p(j) := \sum_{k=1}^{j} k^{-p} \tag{A.6}$$

and $\zeta(3)$ is the Riemann zeta function

$$\zeta(k) := \sum_{j=1}^{\infty}\frac{1}{j^k} \tag{A.7}$$

so that one has

$$g_3(y) = \frac{4}{\pi^{3/2}}G_{0,3}^{3,0}(y^2|0,1,1) \tag{A.8}$$
$$g_4(y) = \frac{16}{3\pi^2}[G_{0,4}^{4,0}(y^2|0,1,1,2) + \mathrm{sign}(y)G_{0,4}^{4,0}(y^2|\tfrac{1}{2},\tfrac{1}{2},\tfrac{3}{2},\tfrac{3}{2})]. \tag{A.9}$$

The graphs of $g_n(y)$ for $n = 1, 2, 3$ and 4 are plotted in figure 1. Note the shape of these curves: for n odd $g_n(y)$ is an even function having a maximum at $y = 0$; while $g_2(y)$ and $g_4(y)$ have their maxima respectively for a negative and a positive value of y in agreement with the sign of the moment $M(2m, 1)$ as given by equation (1.23). We expect this behaviour to be true for all $g_{2m}(y)$.

A.2. Calculation of a determinant

Consider the $n\times n$ determinant $\det[\Gamma(s+i+j]_{i,j=0,\ldots,n-1}$, with s some complex number. A determinant is not changed if we add a constant multiple of a row to another row. Subtract $s+i-1$ times the $(i-1)$th row from the ith row successively for $i = n-1, n-2, \ldots, 1$. Thus the ith row for $1 \leqslant i \leqslant n-1$ changes to

$$\Gamma(s+i+j) - (s+i-1)\Gamma(s+i+j-1)$$
$$= \Gamma(s+i+j-1)[(s+i+j-1) - (s+i-1)] = j\Gamma(s+i+j-1). \tag{A.10}$$

Now the first column $j = 0$ has only one non-zero element, $\Gamma(s)$, in the $(0, 0)$ place, the jth column has a common factor j in the rows $1 \leqslant i \leqslant n-1$, which can be taken out, and the original (i, j) element is changed to $\Gamma(s + i + j - 1)$. Thus

$$\begin{aligned}\det[\Gamma(s+i+j)]_{i,j=0,1,\ldots,n-1} &= \Gamma(s)(n-1)!\det[\Gamma(s+i+j-1)]_{i,j=1,\ldots,n-1}\\ &= \Gamma(s)(n-1)!\det[\Gamma(s+i+j+1)]_{i,j=0,\ldots,n-2}.\end{aligned} \tag{A.11}$$

A recurrence on n now gives

$$\det[\Gamma(s+i+j)]_{i,j=0,\ldots,n-1} = \prod_{j=0}^{n-1} j!\,\Gamma(s+j). \tag{A.12}$$

A.3. Gaussian symplectic and Gaussian orthogonal ensembles

The probability density for the eigenvalues of a matrix taken from the Gaussian symplectic or Gaussian orthogonal ensemble is known to be [1]

$$F_\beta(\boldsymbol{x}) = C_{n\beta}\exp\left(-\sum_{j=1}^{n} x_j^2\right)|\Delta(\boldsymbol{x})|^\beta \qquad \beta = 4 \text{ or } 1. \tag{A.13}$$

The Mellin transforms of the even and odd parts of the probability density $g_{n\beta}(y)$ of its determinant $y = x_1 \ldots x_n$ can again be considered

$$\mathcal{M}^{\pm}_{n\beta}(s) := \int_0^\infty y^{s-1} g^{\pm}_{n\beta}(y)\,\mathrm{d}y \tag{A.14}$$

$$= \tfrac{1}{2}\int_{-\infty}^{\infty} F_\beta(\boldsymbol{x})|x_1\ldots x_n|^{s-1}\varepsilon^{\pm}(x_1\ldots x_n)\,\mathrm{d}x_1\ldots\mathrm{d}x_n \tag{A.15}$$

with $\varepsilon^+(x) = 1$ and $\varepsilon^-(x) = \operatorname{sign}(x)$.

When $\beta = 4$, $\Delta^4(\boldsymbol{x})$ can be expressed as a confluent polynomial alternant [11]

$$\Delta^4(\boldsymbol{x}) = \det[P_{i-1}(x_j), P'_{i-1}(x_j)]_{\substack{i=1,\ldots,2n\\ j=1,\ldots,n}} \tag{A.16}$$

with arbitrary monic polynomials $P_i(x)$. Then an expansion similar to equation (2.7) gives

$$\begin{aligned}\mathcal{M}^{\pm}_{n4}(s) = \tfrac{1}{2}C_{n4}\,2^{-n}\sum_{(i)}\prod_{k=1}^{n}\int_{-\infty}^{\infty}&[P_{i_{2k-1}}(x_k)P'_{i_{2k}}(x_k) - P'_{i_{2k-1}}(x_k)P_{i_{2k}}(x_k)]\\ &\times \mathrm{e}^{-x_k^2}|x_k|^{s-1}\varepsilon^{\pm}(x_k)\,\mathrm{d}x_k\end{aligned} \tag{A.17}$$

$$= \tfrac{1}{2}C_{n4}\,n!\ \mathrm{pf}\,[\Phi^{\pm}_{ij}(4)]_{i,j=0,1,\ldots,2n-1} \tag{A.18}$$

where

$$\Phi^{\pm}_{ij}(4) := \int_{-\infty}^{\infty}\mathrm{e}^{-x^2}|x|^{s-1}\varepsilon^{\pm}(x)[P_i(x)P'_j(x) - P'_i(x)P_j(x)]\,\mathrm{d}x. \tag{A.19}$$

Choosing $P_i(x) = x^i$, one sees that

$$\Phi^{+}_{ij}(4) = \begin{cases}(j-i)\Gamma\left(\dfrac{i+j+s-1}{2}\right) & i+j \text{ odd}\\ 0 & i+j \text{ even}\end{cases} \qquad \operatorname{Re} s > 0 \tag{A.20}$$

$$\Phi^{-}_{ij}(4) = \begin{cases}(j-i)\Gamma\left(\dfrac{i+j+s-1}{2}\right) & i+j \text{ even}\\ 0 & i+j \text{ odd}\end{cases} \qquad \operatorname{Re} s > 0. \tag{A.21}$$

The alternate elements of the matrices $[\Phi^{\pm}_{ij}]_{i,j=0,1,\ldots,2n-1}$ are zero and they can be collected together by a rearrangement of the rows and columns without changing the determinant. Thus one finds that

$$\mathcal{M}^{+}_{n4}(s) = \frac{1}{2}C_{n4}n!\det\left[(2j-2i+1)\Gamma\left(i+j+\frac{s}{2}\right)\right]_{i,j=0,\ldots,n-1} \tag{A.22}$$

$$\mathcal{M}^{-}_{n4}(s) = \frac{1}{2}(-1)^{n/2}C_{n4}n!\,\mathrm{pf}\left[(2j-2i)\Gamma\left(i+j+\frac{s-1}{2}\right)\right]_{i,j=0,1,\ldots,n-1} \times \mathrm{pf}\left[(2j-2i)\Gamma\left(i+j+\frac{s+1}{2}\right)\right]_{i,j=0,1,\ldots,n-1} \qquad n \text{ even} \tag{A.23}$$

$$\mathcal{M}^{-}_{n4}(s) = 0 \qquad n \text{ odd.} \tag{A.24}$$

We did not succeed in finding a compact expression for these determinants valid for general n so as to find their inverse Mellin transforms.

When $\beta = 1$, one can integrate over alternate variables $x_1, x_3, x_5, \ldots,$ for example, to overcome the inconvenience of the absolute value sign of $\Delta(\boldsymbol{x})$ in the integrand. Thus $\mathcal{M}^{\pm}_{n1}(s)$ can again be expressed as a determinant or a Pfaffian depending on the parity of n. For example, when n is even,

$$\mathcal{M}^{\pm}_{n1}(s) = \tfrac{1}{2}C_{n1}n!\det[\Phi^{\pm}_{2i,2j+1}(1)]_{i,j=0,1,\ldots,n/2-1} \tag{A.25}$$

where

$$\Phi^{\pm}_{2i,2j+1}(1) := 2\int_{-\infty<x\leqslant y<\infty} |xy|^{s-1}x^{2i}y^{2j+1}\mathrm{e}^{-x^2-y^2}\varepsilon^{\pm}(xy)\,\mathrm{d}x\,\mathrm{d}y. \tag{A.26}$$

Finding a general expression for these determinants or Pfaffians to study their inverse Mellin transforms is more difficult.

A.4. Complex matrices

If we consider the ensemble of complex matrices, without the Hermitian property, the real and imaginary parts of each matrix element being an independent Gaussian random variable, then the probability density of its complex eigenvalues $\boldsymbol{z} := \{z_j = x_j + \mathrm{i}y_j, 1 \leqslant j \leqslant n\}$, is known [12] for $n = 1, 2, \ldots$

$$F_c(\boldsymbol{z}) = K_c\exp\left(-\sum_{j=1}^{n}|z_j|^2\right)\Delta(z)\Delta(z^*) \qquad K_c^{-1} := \pi^n\prod_{j=1}^{n}j!. \tag{A.27}$$

To find the probability density

$$g_{c,n}(\xi) := \int F(\boldsymbol{z})\delta(\xi - z_1\ldots z_n)\prod_{j=1}^{n}\mathrm{d}x_j\,\mathrm{d}y_j \tag{A.28}$$

of the determinant $\xi := r\mathrm{e}^{\mathrm{i}\theta} = z_1\ldots z_n$, one may use the Fourier series

$$g_{c,n}(\xi) = \sum_{m=-\infty}^{\infty}a_m(r)\mathrm{e}^{\mathrm{i}m\theta} \qquad a_m(r) = \frac{1}{2\pi}\int_0^{2\pi}g_{c,n}(r\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-\mathrm{i}m\theta}\,\mathrm{d}\theta. \tag{A.29}$$

The Mellin transform $\mathcal{A}_m(s)$ of $a_m(r)$ is

$$\mathcal{A}_m(s) := \int_0^{\infty}r^{s-1}a_m(r)\,\mathrm{d}r = \frac{1}{2\pi}\int_0^{\infty}\int_0^{2\pi}[g_{c,n}(r\mathrm{e}^{\mathrm{i}\theta})r^{s-2}\mathrm{e}^{-\mathrm{i}m\theta}]r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \frac{1}{2\pi} \int g_{c,n}(\xi)|\xi|^{s-2} e^{-im \arg \xi} \, d\xi$$

$$= \frac{1}{2\pi} \int F(z) \prod_{j=1}^{n} [|z_j|^{s-2} \exp(-im \arg z_j) \, dx_j \, dy_j]. \tag{A.30}$$

The above expression is again the average value of a function $\Phi(z) = \prod_{j=1}^{n} \phi(z_j)$. Then, the method of section 2 gives

$$\mathcal{A}_m(s) = \frac{1}{2\pi} K_c \, n! \det \left[\int e^{-|z|^2} z^j z^{*k} |z|^{s-2} \exp(-im \arg z) \, dx \, dy \right]_{j,k=0,1,\ldots,n-1}$$

$$= \delta_{m,0} K_c n! \prod_{j=0}^{n-1} \Gamma\left(j + \frac{s}{2}\right). \tag{A.31}$$

This shows that $a_m(r) = 0$ for $m \neq 0$, and $a_0(r)$ is a Meijer G-function,

$$g_{c,n}(\xi) = \left(\pi \prod_{j=0}^{n-1} j!\right)^{-1} G_{0,n}^{n,0}(r^2|0, 1, \ldots, n-1). \tag{A.32}$$

For $n = 1$ and 2 this gives for example, using equations (4.2) and (4.3),

$$g_{c,1}(\xi) = \frac{1}{\pi} G_{0,1}^{1,0}(r^2|0) = \frac{1}{\pi} e^{-r^2} \tag{A.33}$$

$$g_{c,2}(\xi) = \frac{1}{\pi} G_{0,2}^{2,0}(r^2|0, 1) = \frac{2}{\pi} r K_1(2r) \tag{A.34}$$

which can also be verified by a direct calculation.

A.5. Quaternion real matrices

If we consider the ensemble of quaternion real matrices, without the self-dual property, the four real components of each (quaternion real) matrix element being independent Gaussian random variables, then the eigenvalues are complex, $z := \{z_j = x_j + iy_j, 1 \leqslant j \leqslant n\}$, and their probability density is known [15] for $n = 1, 2, \ldots$

$$F_Q(z) = K_Q \exp\left(-\sum_{j=1}^{n} |z_j|^2\right) \prod_{j=1}^{n} |z_j - z_j^*|^2 \prod_{1 \leqslant j < k \leqslant n} |z_j - z_k|^2 |z_j - z_k^*|^2 \tag{A.35}$$

$$K_Q^{-1} := n!(2\pi)^n \prod_{j=1}^{n} (2j-1)!. \tag{A.36}$$

To find the probability density

$$g_{q,n}(\xi) := \int F_Q(z) \delta(\xi - z_1 \ldots z_n) \prod_{j=1}^{n} dx_j \, dy_j \tag{A.37}$$

of the determinant $\xi := r e^{i\theta} = z_1 \ldots z_n$, one may use, as in appendix A.4, the Fourier series

$$g_{q,n}(\xi) = \sum_{m=-\infty}^{\infty} a_m(r) e^{im\theta} \qquad a_m(r) = \frac{1}{2\pi} \int_0^{2\pi} g_{q,n}(r e^{i\theta}) e^{-im\theta} \, d\theta. \tag{A.38}$$

The Mellin transform $\mathcal{A}_m(s)$ of $a_m(r)$ is

$$\mathcal{A}_m(s) := \int_0^{\infty} r^{s-1} a_m(r) \, dr$$

$$= \frac{1}{2\pi}\int_0^\infty\int_0^{2\pi}[g_{q,n}(r\mathrm{e}^{\mathrm{i}\theta})r^{s-2}\mathrm{e}^{-\mathrm{i}m\theta}]r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \frac{1}{2\pi}\int g_{q,n}(\xi)|\xi|^{s-2}\mathrm{e}^{-\mathrm{i}m\arg\xi}\,\mathrm{d}\xi$$

$$= \frac{1}{2\pi}\int F_Q(\boldsymbol{z})\prod_{j=1}^{n}[|z_j|^{s-2}\exp(-\mathrm{i}m\arg z_j)\,\mathrm{d}x_j\,\mathrm{d}y_j]$$

$$= \frac{1}{2\pi}K_Q\int\det[z_j^k,\ z_j^{*k}]_{\substack{j=1,2,\ldots,n\\ k=0,1,\ldots,2n-1}} \qquad \text{(A.39)}$$

$$\times\prod_{j=1}^{n}[(z_j - z_j^*)\mathrm{e}^{-|z|^2}|z_j|^{s-2}\exp(-\mathrm{i}m\arg z_j)\,\mathrm{d}x_j\,\mathrm{d}y_j] \qquad \text{(A.40)}$$

where $\det[z_j^k, z_j^{*k}]$ denotes a $2n\times 2n$ determinant whose $(2j-1)$th column consists of the successive powers of z_j and whose $2j$th column consists of the successive powers of z_j^*, for $j = 1, 2, \ldots, n$.

Expression (A.40) is again the average value of a function $\Phi(\boldsymbol{z}) = \prod_{j=1}^n \phi(z_j)$, the weight being a determinant containing each variable in two columns. An expansion similar to equation (2.7) then gives

$$\mathcal{A}_m(s) = \frac{1}{2\pi}K_Q n!\,\mathrm{pf}[A_{jk}]_{j,k=0,1,\ldots,2n-1} \qquad \text{(A.41)}$$

with

$$A_{jk} := \int (z^j z^{*k} - z^{*j}z^k)(z - z^*)\mathrm{e}^{-|z|^2}|z|^{s-2}\exp(-\mathrm{i}m\arg z)\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^\infty r^{j+k+s-1}\mathrm{e}^{-r^2}r\,\mathrm{d}r\int_0^{2\pi}(\mathrm{e}^{\mathrm{i}(j-k)\theta} - \mathrm{e}^{-\mathrm{i}(j-k)\theta})(\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta})\mathrm{e}^{-\mathrm{i}m\theta}\,\mathrm{d}\theta$$

$$= \pi\Gamma\left(\frac{j+k+s+1}{2}\right)[\delta_{j-k,m-1} - \delta_{j-k,m+1} + \delta_{j-k,-m-1} - \delta_{j-k,-m+1}]. \qquad \text{(A.42)}$$

If $|m| > n+1$, then $A_{n,k} = 0$ for $k = 0, 1, \ldots, 2n-1$, so that $\mathrm{pf}[A_{jk}] = 0 = \mathcal{A}_m(s)$ and $a_m(r) = 0$. Also a careful examination shows that $\mathcal{A}_m(s) = 0$ for n and m both odd, while for any n

$$\mathcal{A}_0(s) = \frac{1}{2\pi}K_Q\ n!(2\pi)^n\prod_{j=0}^{n-1}\Gamma\left(\frac{s+2j+1}{2}\right) \qquad \text{(A.43)}$$

$$\mathcal{A}_{n+1}(s) = \mathcal{A}_{-n-1}(s) = \frac{1}{2\pi}K_Q\ n!(-\pi)^n\prod_{j=0}^{n-1}\Gamma\left(\frac{n+s+2j+1}{2}\right) \qquad \text{(A.44)}$$

showing that $a_{n+1}(r) = a_{-n-1}(r) \neq 0$. Actually $a_m(r) = a_{-m}(r)$, and in general, $a_m(r)\neq 0$ for $|m| \leqslant n+1$, so that $g_{q,n}(\xi)$ is real. $\mathcal{A}_m(s)$ can be calculated with increasing difficulty as $(n+1-|m|)$ increases.

References

[1] See for example, Mehta M L 1991 *Random Matrices* (New York: Academic)

[2] Wigner E P 1965 Distribution laws for the roots of a random Hermitian matrix *Statistical Theories of Spectra: Fluctuations* ed C E Porter (New York: Academic) p 459

[3] Mahoux G, Mehta M L and Normand J-M 1998 Matrices coupled in a chain: II. Spacing functions *J. Phys. A: Math. Gen.* **31** 4457
[4] Luke Y L 1969 *The Special Functions and Their Applications* vol 1 (New York: Academic) ch 5
Bateman H 1953 *Higher Transcendental Functions* vol 1 (New York: McGraw-Hill) 5.3–5.6
Gradsteyn I S and Rizhik I M 1965 *Tables of Integrals Series and Products* (New York: Academic) 9.3
[5] Bateman H 1953 *Higher Transcendental Functions* vol 1 (New York: McGraw-Hill) 1.7
Abramowitch M and Stegun I A 1965 *Handbook of Mathematical Functions* (New York: Dover) 6.3
[6] Widder D V 1971 *Transform Theory* (New York: Academic) 5.7 corollary 7.3a p 109
[7] Bateman H 1954 *Integral Transforms* vol 1 (New York: McGraw-Hill) 6.9 (14)
[8] Luke Y L 1969 *The Special Functions and Their Applications* vol 1 (New York: Academic) 6.5 (8)
Bateman H 1953 *Higher Transcendental Functions* vol 1 (New York: McGraw-Hill) 5.6 (4)
[9] Bateman H 1953 *Higher Transcendental Functions* vol 2 (New York: McGraw-Hill) 7.2.2
Abramowitch M and Stegun I A 1965 *Handbook of Mathematical Functions* (New York: Dover) 9.6
[10] Luke Y L 1969 *The Special Functions and Their Applications* vol 1 (New York: Academic) 5.7 theorem 5 (12)–(15)
[11] See for example, Mehta M L 1989 *Matrix Theory* (Les Ulis: Editions de Physique) ch 7.1
[12] See for example, Mehta M L 1991 *Random Matrices* (New York: Academic) 15.1 (15.1.10) and (15.1.17)
[13] Girko V L 1990 *Theory of Random Determinants (Mathematics and Its Applications, Soviet Series 45)* (Dordrecht: Kluwer Academic)
[14] Nyquist H, Rice S O and Riordan J 1954 The distribution of random determinants *Q. Appl. Math.* **12** 97–104
[15] See for example, Mehta M L 1991 *Random Matrices* (New York: Academic) 15.2 (15.2.10) and (15.2.15)